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Full Length Research Paper

The half way series expansion

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A new formula for mathematics is derived which gives the upper half of the series expansion of the expression $(ay+b)^{2n}$, where n is a natural number. A proof of the new formula is given followed by a simple example to test its accuracy. This formula is helpful whenever n is large.

Key words: Series Expansion, fluid models, Intermolecular Potential model

INTRODUCTION

The usefulness of this formula is not yet known since it is new. Attempt was made to prove beyond reasonable doubt that it is possible to integrate $\int_{\sigma}^{y}\cdots\int_{\sigma}^{y} y^n dy_1\cdots dy_n$

(this is, the function y^n is integrated n times, wherein the region of integration is $[\sigma \quad y]$, n can be any natural number, even if it turns out to be Avogadro's number, as long as the limits of integration is the same throughout. This problem surfaced while attempting to find a mathematically accurate solution to the standard thermodynamic equation for some fluid models. One of the intermolecular potential models attempted was that of the hard sphere which led to Equation 1. Earlier, it was perceived that it is not possible to perform this integration conveniently if n turns out to be a large number. Due to dedication and determination, a general solution was obtained which is true in general (Turay, 2018). That is,

$$\int_{a}^{y}\int_{a}^{y_2}\ldots\int_{a}^{y_n} y_1^n dy_1 dy_2 \ldots dy_n = \frac{n!}{(2n)!}a^n(y-a)^n\sum_{k=0}^{n}\binom{2n}{n-k}\left(\frac{y-a}{a}\right)^k \quad (1)$$

While finding a way of simplifying the aforementioned series, what can best be described as the halfway series expansion was obtained. The result of this series summation is pivotal to help simplify the result. But its other uses cannot be ascertained presently.

SERIES REPRESENTATION OF THE HALFWAY EXPANSION

Consider the function:

$$f(y) = (ay+b)^{2n} \quad (2)$$

The right hand side can be expanded and obtain (Burton, 2007; Guichar, 2017; Kalman, 1993):

$$(ay+b)^{2n} = (ay)^{2n}\binom{2n}{0} + (ay)^{2n-1}b\binom{2n}{1} + \cdots + (ay)^n b^n\binom{2n}{n} + \cdots + (ay)b^{2n-1}\binom{2n}{2n-1} + b^{2n}\binom{2n}{2n}$$

In series notation $(ay+b)^{2n} = \sum_{r=0}^{2n}\binom{2n}{r}(ay)^{2n-r}b^r$ (Burton,

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2007; Guichar, 2017), the upper halfway in the series expansion of the function $f(y)$ is given by

$$hf(y) = (ay)^n b^n \binom{2n}{n} + \cdots + (ay)b^{2n-1}\binom{2n}{2n-1} + b^{2n}\binom{2n}{2n} \quad (3)$$

$$hf(y) = (ay)^n b^n \left[\binom{2n}{n} + \cdots + \left(\frac{b}{ay}\right)^{n-1}\binom{2n}{2n-1} + \left(\frac{b}{ay}\right)^n\binom{2n}{2n} \right] \quad (4)$$

In series notation (Burton, 2007):

$$hf(y) = (ay)^n b^n \left[\binom{2n}{n} + \sum_{n-r=1}^{n} \left(\frac{b}{ay}\right)^{n-r}\binom{2n}{2n-r} \right]$$

This can be rewritten as

$$hf(y) = (ay)^n b^n \left[\sum_{n-r=0}^{n} \left(\frac{b}{ay}\right)^{n-r}\binom{2n}{2n-r} \right]$$

Now let $n - r = k$

Substituting this into the aforementioned equation gives (Guichar, 2017)

$$hf(y) = (ay)^n b^n \left[\sum_{k=0}^{n} \left(\frac{b}{ay}\right)^{k}\binom{2n}{n+k} \right] \quad (5)$$

Thus, in series notation, the upper halfway expansion of $f(y) = (ay+b)^{2n}$ is given by (Burton, 2007; Guichar, 2017):

$$hf(y) = (ay)^n b^n \left[\sum_{k=0}^{n} \left(\frac{b}{ay}\right)^{k}\binom{2n}{n-k} \right] \quad (6)$$

This holds since $\binom{2n}{n+k} = \binom{2n}{n-k}$

RIGOROUS PROOF OF THE SUMMATION OF SERIES

Furthermore, let us introduce the function:

$$h_1 f(y) = \sum_{k=0}^{n} \left(\frac{b}{ay}\right)^{k}\binom{2n}{n-k} \quad (7)$$

Let us further make the substitution:

$$e^{2x} = \frac{b}{ay} \Rightarrow 2x = \ln\left(\frac{b}{ay}\right) \Rightarrow x = \frac{1}{2}\ln\left(\frac{b}{ay}\right)$$

This reduces the function $h_1 f(y)$ to a new function $h_1 f(y) = H_1(x)$ (Wrede and Speigel, 2010):

where $$H_1(x) = \sum_{k=0}^{n}\binom{2n}{n-k}e^{2kx} = \sum_{k=0}^{n}[\cosh 2kx + \sinh 2kx]\binom{2n}{n-k} \quad (8)$$

$$= \sum_{k=0}^{n}\cosh 2kx\binom{2n}{n-k} + \sum_{k=0}^{n}\sinh 2kx\binom{2n}{n-k}$$

Introducing other new functions called:

$$J(x) = \sum_{k=0}^{n}\binom{2n}{n-k}\cosh 2kx \text{ and } \phi(x) = \sum_{k=0}^{n}\binom{2n}{n-k}\sinh 2kx$$

This gives

$$H_1(x) = J(x) + \phi(x) \quad (9)$$

First attempt was made to prove that

$$J(x) = \sum_{k=0}^{n}\binom{2n}{n-k}\cosh 2kx = \frac{1}{2}(e^x + e^{-x})^{2n} + \frac{1}{2}\binom{2n}{n} \quad (10)$$

This was started by finding an expansion for the function

$$= \binom{2n}{0}e^{2nx} + \binom{2n}{2n}e^{-2nx} + \binom{2n}{1}e^{2(n-1)x} + \binom{2n}{2n-1}e^{-2(n-1)x} + \cdots + \binom{2n}{p}e^{2(n-p)x} + \binom{2n}{2n-p}e^{-2(n-p)x} + \cdots + \binom{2n}{n}$$

$$(e^x + e^{-x})^{2n} = \binom{2n}{0}e^{2nx} + \binom{2n}{1}e^{2(n-1)x} + \cdots + \binom{2n}{p}e^{2(n-p)x} + \cdots + \binom{2n}{2n-p}e^{-2(n-p)x} + \cdots + \binom{2n}{2n-1}e^{-2(n-1)x} + \binom{2n}{2n}e^{-2nx}$$

The first and last term, the second and second to last, etc., were taken, then it was generalized by obtaining the pth term and the $(2n-p)^{th}$ term (Burton, 2007).

Making use of the fact that $\binom{2n}{p} = \binom{2n}{2n-p}$, $p = 0, \cdots, 2n$ (Burton, 2007; Guichar, 2017), it can be rewritten as:

$$\binom{2n}{p}e^{2(n-p)x} + \binom{2n}{2n-p}e^{-2(n-p)x} = \binom{2n}{p}\left[e^{2(n-p)x} + e^{-2(n-p)x}\right] = 2\binom{2n}{p}\cosh 2(n-p)x$$

With the substitution $p = n - k$, it was clearly see that (Swokowski, 1979):

$$\binom{2n}{p}\cosh 2(n-p)x = \binom{2n}{n-k}\cosh 2kx,$$

ioThis expression holds for every value of p and so it trivially holds for every k, since $p = n - k$.

Thus $(e^x + e^{-x})^{2n} = \binom{2n}{n} + 2\sum_{k=1}^{n}\binom{2n}{n-k}\cosh 2kx$

By simplifying, we obtain:

$$\sum_{k=1}^{n}\binom{2n}{n-k}\cosh 2kx = \frac{1}{2}(e^x + e^{-x})^{2n} - \frac{1}{2}\binom{2n}{n}$$

By taking into consideration the case $k = 0$, the result follows:

$$\sum_{k=0}^{n}\binom{2n}{n-k}\cosh 2kx = \frac{1}{2}(e^x + e^{-x})^{2n} + \frac{1}{2}\binom{2n}{n}$$

Then attention given to finding an expression for
$\phi(x) = \sum_{r=0}^{n}\binom{2n}{n-r}\sinh 2rx$ Clearly

$\sum_{r=0}^{n}\binom{2n}{n-r}\sinh 2kx = \sum_{r=1}^{n}\binom{2n}{n-r}\sinh 2rx$ since $\sinh(0) = 0$

So it was observed that:

$$\phi(x) = \sum_{r=1}^{n}\binom{2n}{n-r}\sinh 2rx$$

An important formula is (Burton, 2007):

$\binom{\alpha+1}{\beta+1} = \frac{(\alpha+1)!}{(\alpha-\beta)!(\beta+1)!} = \frac{(\alpha+1)\alpha!}{(\alpha-\beta)!(\beta+1)(\beta)!} = \frac{\alpha+1}{\beta+1}\frac{(\alpha)!}{(\alpha-\beta)!(\beta)!} = \frac{\alpha+1}{\beta+1}\binom{\alpha}{\beta}$

That is $\binom{\alpha+1}{\beta+1} = \frac{\alpha+1}{\beta+1}\binom{\alpha}{\beta}$

Applying this formula, another relationship that was later found helpful was arrived at (Swokowski, 1979).

This   helpful   relationship   is
$\frac{n-r+1}{2n+1}\binom{2n+1}{n-r+1} = \binom{2n}{n-r}$

Thus $\phi(x) = \sum_{r=1}^{n}\frac{n-r+1}{2n+1}\binom{2n+1}{n-r+1}\sinh 2rx$ $\Rightarrow$

(see Swokowski, 1979)

Another result that was found useful is:

$\binom{\alpha}{\beta} + \binom{\alpha}{\beta+1} = \binom{\alpha+1}{\beta+1}$ (This is known as Pascal's Identity (Burton, 2007):

Proof

$\binom{\alpha}{\beta} + \binom{\alpha}{\beta+1} = \frac{(\alpha)!}{(\alpha-\beta)!(\beta)!} + \frac{(\alpha)!}{(\alpha-\beta-1)!(\beta+1)!}$

$= \frac{(\alpha)!}{(\alpha-\beta)(\alpha-\beta-1)!(\beta)!} + \frac{(\alpha)!}{(\alpha-\beta-1)!(\beta+1)(\beta)!}$

$= \left[\frac{(\alpha)!}{(\alpha-\beta-1)!(\beta)!}\right]\left[\frac{1}{(\alpha-\beta)} + \frac{1}{(\beta+1)}\right]$

$= \left[\frac{(\alpha)!}{(\alpha-\beta-1)!(\beta)!}\right]\left[\frac{(\beta+1)+(\alpha-\beta)}{(\alpha-\beta)(\beta+1)}\right]$

$= \left[\frac{(\alpha)!}{(\alpha-\beta-1)!(\beta)!}\right]\left[\frac{1+\alpha}{(\alpha-\beta)(\beta+1)}\right]$

$= \frac{(\alpha+1)(\alpha)!}{(\alpha-\beta)(\alpha-\beta-1)!(\beta+1)(\beta)!}$

$= \frac{(\alpha+1)!}{(\alpha-\beta)!(\beta+1)!}$

$= \binom{\alpha+1}{\beta+1}$

This   clearly   makes   the   relationship
$\binom{2n+1}{n-r+1} = \binom{2n}{n-r} + \binom{2n}{n-r+1}$ valid. By substituting this into the present equation, we arrive at:

$$(2n+1)\phi(x) = (n+1)\sum_{r=1}^{n}\left[\binom{2n}{n-r} + \binom{2n}{n-r+1}\right]\sinh 2rx - \sum_{r=1}^{n}r\left[\binom{2n}{n-r} + \binom{2n}{n-r+1}\right]\sinh 2rx$$

$$= (n+1)\sum_{r=1}^{n}\binom{2n}{n-r}\sinh 2rx + (n+1)\sum_{r=1}^{n}\binom{2n}{n-r+1}\sinh 2rx - \sum_{r=1}^{n}r\binom{2n}{n-r}\sinh 2rx - \sum_{r=1}^{n}r\binom{2n}{n-r+1}\sinh 2rx$$

$$= (n+1)\phi(x) + (n+1)\sum_{r=1}^{n}\binom{2n}{n-r+1}\sinh 2rx - \sum_{r=1}^{n}r\binom{2n}{n-r}\sinh 2rx - \sum_{r=1}^{n}r\binom{2n}{n-r+1}\sinh 2rx$$

Thus

$$n\phi(x) = (n+1)\sum_{r=1}^{n}\binom{2n}{n-r+1}\sinh 2rx - \sum_{r=1}^{n}r\binom{2n}{n-r}\sinh 2rx - \sum_{r=1}^{n}r\binom{2n}{n-r+1}\sinh 2rx \quad (11)$$

Let $\quad Y_A = \sum_{r=1}^{n}\binom{2n}{n-r+1}\sinh 2rx$

$Y_B = \sum_{r=1}^{n}r\binom{2n}{n-r}\sinh 2rx$

$Y_C = \sum_{r=1}^{n}r\binom{2n}{n-r}\sinh 2rx$

$\Rightarrow n\phi(x) = (n+1)Y_A - Y_B - Y_C \quad (12)$

Now $Y_A = \sum_{r=1}^{n}\binom{2n}{n-r+1}\sinh 2rx \quad (13)$

Let us make the substitution $r = p+1$ (Wrede and Speigel, 2010):

Thus $\quad Y_A = \sum_{p=0}^{n-1}\binom{2n}{n-p}\sinh 2(p+1)x$

$Y_A = \sum_{p=0}^{n-1}\binom{2n}{n-p}[\sinh 2px\cosh 2x + \cosh 2px\sinh 2x]$

$= \left[\sum_{p=0}^{n-1}\binom{2n}{n-p}\sinh 2px\right]\cosh 2x + \left[\sum_{p=0}^{n-1}\binom{2n}{n-p}\cosh 2px\right]\sinh 2x$

$= [\phi(x) - \sinh 2nx]\cosh 2x + [J(x) - \cosh 2nx]\sinh 2x$

$= \cosh 2x\phi(x) - \cosh 2x\sinh 2nx + \sinh 2xJ(x) - \sinh 2x\cosh 2nx$

$Y_A = \cosh 2x\phi(x) + \sinh 2xJ(x) - \sinh 2(n+1)x \quad (14)$

$Y_B = \sum_{r=1}^{n}r\binom{2n}{n-r}\sinh 2rx \quad (15)$

$= \frac{1}{2}\frac{d}{dx}\sum_{r=1}^{n}\binom{2n}{n-r}\cosh 2rx$

$= \frac{1}{2}\frac{d}{dx}\left[J(x) - \binom{2n}{n}\right]$

$Y_B = \frac{1}{2}J'(x) \quad (16)$

$Y_C = \sum_{r=1}^{n}r\binom{2n}{n-r+1}\sinh 2rx \quad (17)$

$= \frac{1}{2}\frac{d}{dx}\sum_{r=1}^{n}\binom{2n}{n-r+1}\cosh 2rx$

With the substitution $r = p+1$, we arrive at

$Y_C = \frac{1}{2}\frac{d}{dx}\sum_{p=0}^{n-1}\binom{2n}{n-p}\cosh 2(p+1)x$

$= \frac{1}{2}\frac{d}{dx}\left[\sum_{p=0}^{n-1}\binom{2n}{n-p}\cosh 2px\cosh 2x + \sinh 2px\sinh 2x\right]$

$= \frac{1}{2}\frac{d}{dx}\left[\cosh 2x\sum_{p=0}^{n-1}\binom{2n}{n-p}\cosh 2px + \sinh 2x\sum_{p=0}^{n-1}\binom{2n}{n-p}\sinh 2px\right]$

$= \frac{1}{2}\frac{d}{dx}[(J(x) - \cosh 2nx)\cosh 2x + (\phi(x) - \sinh 2nx)\sinh 2x]$

$= \frac{1}{2}\left[\begin{array}{l}(J'(x) - 2n\sinh 2nx)\cosh 2x + (J(x) - \cosh 2nx)2\sinh 2x \\ + (\phi'(x) - 2n\cosh 2nx)\sinh 2x + (\phi(x) - \sinh 2nx)2\cosh 2x\end{array}\right]$

$= \frac{1}{2}J'(x)\cosh 2x - (n+1)[\sinh 2nx\cosh 2x + \cosh 2nx\sinh 2x] + J(x)\sinh 2x + \frac{1}{2}\phi'(x)\sinh 2x + \phi(x)\cosh 2x$

$Y_C = \frac{1}{2}J'(x)\cosh 2x - (n+1)\sinh 2(n+1)x + J(x)\sinh 2x + \frac{1}{2}\phi'(x)\sinh 2x + \phi(x)\cosh 2x \quad (18)$

It is clear that

$$n\phi(x) = (n+1)Y_A - Y_B - Y_C$$

So

$$n\phi(x) = \left(\begin{array}{l}(n+1)\cosh 2x\phi(x) + (n+1)\sinh 2xJ(x) - (n+n)\sinh 2(n+1)x - \frac{1}{2}J'(x) \\ -\left[\frac{1}{2}J'(x)\cosh 2x + j(x)\sinh 2x + \frac{1}{2}\phi'(x)\sinh 2n + \phi(x)\cosh 2x - (n+1)\sinh 2(n+1)x\right]\end{array}\right)$$

Some of the terms cancel out and we are left with:

$$n\phi(x) = n\cosh 2x\phi(x) + n\sinh 2xJ(x) - \frac{1}{2}J'(x)(1+\cosh 2x) - \frac{1}{2}\phi'(x)\sinh 2x \quad (19)$$

Digress (Swokowski, 1979; Wrede and Speigel, 2010):

$$J(x) = \sum_{r=0}^{n} \binom{2n}{n-r} \cosh 2rx = \frac{1}{2}\left[\left(e^x + e^{-x}\right)^{2n} + \binom{2n}{n} \right]$$

$$J'(x) = n\left(e^x + e^{-x}\right)^{2n-1}\left(e^x - e^{-x}\right)$$

$$1 + \cosh 2x = 2\cosh^2 x$$

$$n\sinh 2xJ(x) - \frac{1}{2}J'(x)[1+\cosh 2x] = n\cosh x \sinh x\left[\left(e^x+e^{-x}\right)^{2n}+\binom{2n}{n}\right] - n\left(e^x+e^{-x}\right)^{2n-1}\left(e^x-e^{-x}\right)\cosh^2 x$$

$$= n\cosh x \sinh x\left[\left(e^x+e^{-x}\right)^{2n}+\binom{2n}{n}\right] - n\cosh x\sinh x\left(e^x+e^{-x}\right)^{2n}$$

$$= n\cosh x \sinh x \binom{2n}{n}$$

$$= \frac{1}{2}n\sinh 2x \binom{2n}{n}$$

Making the above substitutions gives us

$$n\phi(x) = n\cosh 2x\phi(x) - \frac{1}{2}\phi'(x)\sinh 2x + \frac{1}{2}n\sinh 2x\binom{2n}{n}$$

$$\Rightarrow$$

$$\frac{1}{2}\phi'(x)\sinh 2x + n\phi(x)[1-\cosh 2x] = \frac{1}{2}n\sinh 2x\binom{2n}{n}$$

$$\Rightarrow$$

$$\phi'(x)\sinh x\cosh x - 2n\phi(x)\sinh^2 x = n\sinh x\cosh x\binom{2n}{n}$$

$$\Rightarrow \phi'(x)\cosh x - 2n\phi(x)\sinh x = n\cosh x\binom{2n}{n}$$

Thus

$$\phi'(x) - 2n\phi(x)\tanh x = n\binom{2n}{n} \tag{20}$$

This first order differential equation can be solved by simply finding an integrating factor. Let (Bronson and Bredensteine, 2003):

$$P = -2n\tanh x$$

where R is the integrating factor, we use the known formula $R = e^{\int Pdx}$

Now $\int Pdx = \int -2n\tanh x\, dx = -2n\ln\cosh x$

So   that R becomes
$$R = e^{-2n\ln\cosh x} = \left[e^{\ln\cosh x}\right]^{(-2n)} = (\cosh x)^{-2n} = \left(\operatorname{sech}^2 x\right)^n$$
The   solution   thus   becomes

$$\frac{d}{dx}\left[\phi(x)\left(\operatorname{sech}^2 x\right)^n\right] = n\binom{2n}{n}\left(\operatorname{sech}^2 x\right)^n$$

Thus

$$\phi(x) = n\binom{2n}{n}\left(\cosh^2 x\right)^n \int_0^x \left(\operatorname{sech}^2 t\right)^n dt \tag{21}$$

where t is a dummy variable and the lower limit of integration is 0 because $\phi(0) = 0$.

THE NEW FORMULA

Going back to Equation 9,

$$H_1(x) = J(x) + \phi(x)$$

$$H_1(x) = \frac{1}{2}(e^x+e^{-x})^{2n} + \frac{1}{2}\binom{2n}{n} + n\binom{2n}{n}\left(\cosh^2 x\right)^n\int_0^x\left(\operatorname{sech}^2 t\right)^n dt \tag{22}$$

Making the substitution $x = \frac{1}{2}\ln\left(\frac{b}{ay}\right)$

It is seen that $e^x + e^{-x} = \sqrt{\frac{b}{ay}} + \sqrt{\frac{ay}{b}}$

So

$$h_1 f(y) = \frac{1}{2}\left(\frac{b+ay}{\sqrt{aby}}\right)^{2n} + \frac{1}{2}\binom{2n}{n} + n\binom{2n}{n}\left(\frac{ay+b}{2\sqrt{aby}}\right)^{2n}\int_0^x\left(\operatorname{sech}^2 t\right)^n dt \tag{23}$$

Thus the halfway summation

$$hf(y) = (ay)^n b^n\left[h_1(y)\right] \tag{24}$$

$$hf(y) = (ay)^n b^n\left[\frac{1}{2}\left(\frac{b+ay}{\sqrt{aby}}\right)^{2n} + \frac{1}{2}\binom{2n}{n} + n\binom{2n}{n}\left(\frac{ay+b}{2\sqrt{aby}}\right)^{2n}\int_0^x\left(\operatorname{sech}^2 t\right)^n dt\right] \tag{25}$$

SIMPLE ILLUSTRATION TO TEST RESULT

Let us test the result for a small value of n because we can easily cope with it by hand. Say for instance $n = 2$, the present formula simplifies to

$$hf(y) = (ay)^2 b^2\left[\frac{1}{2}\left(\frac{ay+b}{\sqrt{aby}}\right)^4 + \frac{1}{2}\binom{4}{2} + 2\binom{4}{2}\left(\frac{ay+b}{2\sqrt{aby}}\right)^4\int_0^x\left(\operatorname{sech}^2 t\right)^2 dt\right]$$

$$= (ay)^2 b^2 \left[\frac{1}{2} \frac{(ay+b)^4}{(aby)^2} + \frac{1}{2}(6) + 2(6) \frac{(ay+b)^4}{16(aby)^2} \int_0^x (\mathrm{sec}h^2 t)^2 \, dt \right]$$

$$= (ay)^2 b^2 \left[\frac{1}{2} \frac{(ay+b)^4}{(aby)^2} + 3 + \frac{3}{4} \frac{(ay+b)^4}{(aby)^2} \int_0^x (\mathrm{sec}h^2 t)^2 \, dt \right]$$

Consider $\displaystyle\int_0^x (\mathrm{sec}h^2 t) dt = \int_0^x (\mathrm{sec}h^2 t)(\mathrm{sec}h^2 t) dt$

$$= \int_0^x \mathrm{sec}h^2 t \, d(\tanh t)$$

$$= \int_0^x (1 - \tanh^2 t) \, d(\tanh t)$$

$$= \left[\tanh t - \frac{1}{3}\tanh^3 t \right]_0^x$$

$$= \tanh x - \frac{1}{3}\tanh^3 x$$

$$= \frac{\sqrt{\dfrac{b}{ay}} - \sqrt{\dfrac{ay}{b}}}{\sqrt{\dfrac{b}{ay}} + \sqrt{\dfrac{ay}{b}}} - \frac{1}{3}\left(\frac{\sqrt{\dfrac{b}{ay}} - \sqrt{\dfrac{ay}{b}}}{\sqrt{\dfrac{b}{ay}} + \sqrt{\dfrac{ay}{b}}} \right)^3$$

$$= \frac{b-ay}{b+ay} - \frac{1}{3}\left(\frac{b-ay}{b+ay} \right)^3$$

Thus, we have (Swokowski, 1979; Bronson and Bredensteiner, 2003):

$$hf(y) = (ay)^2 b^2 \left[\frac{1}{2}\frac{(ay+b)^4}{(aby)^2} + 3 + \frac{3}{4}\frac{(ay+b)^4}{(aby)^2}\left[\frac{b-ay}{b+ay} - \frac{1}{3}\left(\frac{b-ay}{b+ay}\right)^3 \right] \right]$$

$$hf(y) = \frac{1}{2}(ay+b)^4 + 3a^2 b^2 y^2 + \frac{3}{4}(ay+b)^4\left[\frac{b-ay}{b+ay} - \frac{1}{3}\left(\frac{b-ay}{b+ay}\right)^3 \right]$$

$$hf(y) = \frac{1}{2}(ay+b)^4 + 3a^2 b^2 y^2 + \frac{3}{4}(ay+b)^3(b-ay) - \frac{1}{4}(b+ay)(b-ay)^3$$

$$= \frac{1}{2}(ay+b)^4 + 3a^2 b^2 y^2 + \frac{1}{4}(b+ay)(b-ay)\left[3(b+ay)^2 - (b-ay)^2 \right]$$

$$= \frac{1}{2}(ay+b)^4 + 3a^2 b^2 y^2 + \frac{1}{4}\left(b^2 - (ay)^2 \right)\left[2b^2 + 8aby + 2a^2 y^2 \right]$$

$$= \frac{1}{2}(ay+b)^4 + 3a^2 b^2 y^2 + \frac{1}{2}\left(b^2 - (ay)^2 \right)\left(b^2 + 4aby + a^2 y^2 \right)$$

$$= \frac{1}{2}\left[(ay)^4 + 4(ay)^3 b + 6(ay)^2 b^2 + 4(ay)b^3 + b^4 \right] + 3a^2 b^2 y^2 + \frac{1}{2}\left[b^4 + 4ab^3 y + a^2 b^2 y^2 - a^2 b^2 y^2 - 4a^3 by^3 - a^4 y^4 \right]$$

$$= \frac{1}{2}\left[6a^2 b^2 y^2 + 8ab^3 y + 2b^4 \right] + 3a^2 b^2 y^2$$

$$= 3a^2 b^2 y^2 + 4ab^3 y + b^4 + 3a^2 b^2 y^2$$

$$= 6a^2 b^2 y^2 + 4ab^3 y + b^4$$

This is the exact expression for the upper half way expansion of $(ay+b)^4$.

OTHER APPLICATIONS OF THE HALFWAY FORMULA

A series for $(1-q)^{-n}$ will be built. If we happen to take the painful task of extending the Pascal's triangle by inducing the terms underneath which turns out to be the lower half of another triangle, so that the combination of these two gives a square (Figure 1). Following the direction of the arrows, we see that, the first arrow depicts the coefficient of the first few terms of the expansion of the series $(1-q)^{-1}$, while the second gives the first few terms of the coefficients of the expansion of $(1-q)^{-2}$, etc. From Figure 1, a series is built (Kalman, 1993):

$$(1-q)^{-n} = \sum_{k=0}^{\infty} \binom{n-1+k}{n-1} q^k \quad n \geq 1 \ (n \text{ is a natural}$$

number) $\quad |q| < 1$ (26)

From Equation 26, we obtain the series:

$$\left(1 - \tfrac{1}{q}\right)^{-n} = \sum_{k=0}^{\infty} \binom{n-1+k}{n-1} q^{-k} \quad |q| > 1. \tag{27}$$

Furthermore, from Equation 27, if we take the sum of the first $n+1$ terms of the series and then multiply this series by q^n, a series will be obtained whose first $n+1$ coefficients are the same as those in Equation 26, but with the order of its powers reversed (in descending order). The new series described so far can be expressed by the formula:

$$\varphi\left[q^n (1-\tfrac{1}{q})^{-n} \right] = \sum_{k=0}^{n} \binom{n-1+k}{n-1} q^{n-k} \tag{28}$$

Now in other to see the usefulness of Equation 28, let us extract from Equation 1, the expression:

$$\sum_{k=0}^{n} \binom{2n}{n-k}\left(\frac{y-\sigma}{\sigma} \right)^k = \sum_{k=0}^{n} \binom{2n}{n-k}\left(\frac{y}{\sigma} - 1 \right)^k$$

Substituting this in Equation 28, gives us:

$$\sum_{k=0}^{n} \binom{2n}{n-k}\left(\frac{y}{\sigma} - 1 \right)^k = \sum_{k=0}^{n} \binom{n-1+k}{n-1}\left(\frac{y}{\sigma} \right)^{n-k} \tag{29}$$

Figure 1. The induced Pascal's square.

A proof of Equation 29 is yet to be obtained; but from simple illustrations it is true. Its accuracy for small values of n has been tested. Well say for instance $n=3$ and $q=\dfrac{y}{\sigma}$

Right hand side will be (Guichard, 2017):

$$\sum_{k=0}^{3}\binom{2+k}{2}q^{3-k}=\binom{2}{2}q^3+\binom{3}{2}q^2+\binom{4}{2}q+\binom{5}{2}=q^3+3q^2+6q+10$$

For the left hand side (Burton, 2007), we have:

$$\sum_{k=0}^{3}\binom{6}{3-k}(q-1)^k=\binom{6}{3}+\binom{6}{2}(q-1)+\binom{6}{1}(q-1)^2+\binom{6}{0}(q-1)^3$$
$$=20+15(q-1)+6(q^2-2q+1)+(q^3-3q^2+3q-1)$$
$$=(20-15+6-1)+(15-12+3)q+(6-3)q^2+q^3$$
$$=10+6q+3q^2+q^3$$

So left hand side = right hand side.

From Equation 29, if the substitution $q=\dfrac{y}{\sigma}$ is made, then we obtain:

$$\sum_{k=0}^{n}\binom{2n}{n-k}(q-1)^k=\sum_{k=0}^{n}\binom{n-1+k}{n-1}(q)^{n-k}$$

When $q-1=s$ is substituted, the equation reduces to:

$$\sum_{k=0}^{n}\binom{2n}{n-k}s^k=\sum_{k=0}^{n}\binom{n-1+k}{n-1}(s+1)^{n-k} \tag{30}$$

This new Equation 30 is clearly the halfway series expansion of $(s+1)^{2n}$, that is

$$h(s+1)^{2n}=\sum_{k=0}^{n}\binom{2n}{n-k}s^k=\sum_{k=0}^{n}\binom{n-1+k}{n-1}(s+1)^{n-k} \tag{31}$$

A RELATIONSHIP DERIVED FROM THE INDUCED PASCAL'S SQUARE

The formula

$$\sum_{k=0}^{m}\binom{n-1+k}{n-1}=\binom{n+m}{n} \tag{32}$$

This is valid for all $m\geq 1$, also may be found useful (Enochs, 2004) and is also deducible from Figure 1.

CONFLICT OF INTERESTS

The authors have not declared any conflict of interests.

REFERENCES

Bronson R, Bredensteiner EJ (2003).   Schaum's Series Differential Equation. McGraw Hill, New York.

Burton DM (2007). Elementary Number Theory (sixth Edition) McGraw Hill, New York.

Enochs EE (2004). Binomial Coefficients. Boletin de la Association Matematica Venezolana 11(1):17.

Guichar D (2017). An Introduction to Combinatorics and Graph Theory. http:www.whitman.edu/mathematics/cgt_online/cgt.pdf

Kalman D (1993). Six ways to sum a series. The College Mathematics Journal 24(5):402-421.

Swokowski EW (1979). Calculus with Analytic Geometry (Second Edition). Prindle, Weber & Schmidt, Boston, Massachusetts.

Turay AI (2018). Equations of State obtained via Thermodynamic Considerations. Tertiary Education Commission Journal, Sierra Leone.

Wrede R, Speigel MR (2010). Schaum's Series Advance Calculus. Third edition. McGraw Hill, New York.

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Maximizing profit in the poultry farming sector: An application of the robust linear programming technique

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Managing the limited resources of money, time and space poses a real problem in the livestock business in general and the poultry sector in particular. The stochastic linear programming with recourse will permit the formulation of a mathematical model that could easily be solved in order to attain the objective of maximizing profit but this will be hindered by the presence of some parameters like demand and supply which have no predetermined probability distributions. In an attempt to address the problem of unavailable probability distribution functions, this research is aimed at proposing a robust linear programming model which can handle problems and others in similar circumstances with the need of probability distributions.The robust model thus constructed is based on a modification of the stochastic model by Soyster. An application of this model on a real life data produced results showing an increase in profit made by a local poultry farmer from 241,485 FCFA to 362,580FCFA representing an increase in profit of more than 50% in over that obtained when using the ordinary Linear Programming Technique. The belief therefore is that, if models like this are implemented, not only would the livelihood of the poultry farmers be improved, but it will go a long way to better the economy and satisfy the ever increasing need of poultry products by the communities.

Key words: Maximization, livestock, stochastic, robust, linear programming, optimization.

INTRODUCTION

Livestock systems occupy about 30% of the planet's ice-free terrestrial surface area (Steinfeld et al., 2006) and are a significant global asset with a value of at least $1.4 trillion. The livestock sector is increasingly organized in long market chains that employ at least 1.3 billion people globally and directly support the livelihoods of 600 million poor smallholder farmers in the developing world (Thornton et al., 2006). Keeping livestock is an important risk reduction strategy for vulnerable communities, and livestock are important providers of nutrients and traction for growing crops in smallholder systems. Livestock products contribute 17% to kilocalorie consumption and 33% to protein consumption globally (Rosegrant et al., 2009). From the World Bank 2009 report, livestock

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systems have both positive and negative effects on the natural resource base, public health, social equity and economic growth. Despite this, livestock is one of the fastest growing agricultural subsectors in developing countries. Its share of agricultural GDP is already 33% and is quickly increasing. This growth is driven by the rapidly increasing demand for livestock products; this demand being driven by population growth, urbanization and increasing incomes in developing countries (Delgado, 2005) while traditional livestock systems contribute to the livelihoods of 70% of the world's rural poor, increasingly the emerging large-scale operations with sophisticated technology and international trade cater for the rapidly growing markets for meat, milk and eggs. The AGA's program on animal production focuses primarily on small-scale dairying, small-medium scale poultry and, to a lesser extent, on small ruminant systems which can make a significant contribution to improved livelihoods and local economic development. This will be achieved through the provision of topical information, guidance and technical support to farmers.

Problem statement

In the developing world where more than 600 million small holder farmers and in Cameroon where about 80% of the population depends on agriculture for livelihood (Nji, 2004), with millions relying on small-medium scale poultry, there is need to provide technical support to these farmers in order to boost production. There has been rapid global expansion of consumption of poultry products and which is expected to continue to grow. Increasing productivity, especially in the small to medium scale production systems, is currently constrained by lack of skills, knowledge and appropriate technologies compounded by the inability to manage limited resources of money, space and time, especially as demand and supply cannot be determined in advance.

The production and commercialization of livestock like every economic activity requires the rational use of limited resources of money, space and time available at any given time with the objective of maximizing profits. The Simplex methods of Dantzig (1955) would have been a better tool to solve the problem if the parameters were static and could be modelled by Linear Programming. This is not the case with the livestock sector especially with the small-medium scale poultry farming where most parameters such as demand and supply cannot be determined ahead of time. The significance of uncertainty has prompted a number of works addressing random parameters in tactical level supply chain planning involving production and distribution of products in different sector of businesses; examples can be seen in Alidaee and Kochenberger (2005), Cooper and LeBlanc (1997), Powell and Topaloglu (2003), Yu and Li (2000), and Van Landeghem and Vanmaele (2002). On the other hand, Mustafa (2005) and Ziemba and Vickson (1975) suggested that, under some assumed probability distribution of the uncertain parameters, a stochastic model could handle the optimization. What happens in the absence of any assumed probability distribution of these parameters? This is the concern of these researchers.

Therefore, the objective of this research was to propose a robust linear programming (RLP) model that could be used to handle optimization problems in the livestock business sector in general and in poultry farming in particular in the absence of probability distribution of the uncertain parameters involved. In the case of the small-medium scale poultry farming, the main concern shall be to determine how many species of each type and age of birds to be kept at a particular period in order to maximize the benefit without violating the constraints of money, time, and available space while using the robust linear programming model.

BACKGROUND AND RELATED WORK

Most often in a linear program problem, the parameters are random variables whose probability distributions cannot be determined ahead of time within each production period. This has thus left most decision makers in the field

Stochastic programming (SP) and Robust Optimization (RO) are the main alternative techniques to deal with uncertain data both in a single period and in a multi-period decision making process. The main difficulty associated with the former is the need to provide the probability distribution functions of the underlying stochastic parameters. This requirement creates a heavy burden on the user because in many real world situations, such information is unavailable or hard to obtain (Birge and Louveaux, 2011; Ruszczinski and Shapiro, 2003). On the other hand, RO addresses the uncertain nature of the problem without making specific assumptions on probability distributions. The goal is to seek solutions which are insensitive to variations of certain data of the problem without destroying all or part of the potential optimum. This is made possible by using the technique of Soyster (1973) or Minoux (2009, 2010). Other reflections on treating problems on uncertainty can be seen in Dupacova (1998).

Soyster (1973) introduced uncertainty as a column in the linear program. It showed that a linear programming problem can be written in the form:

$$Q(x) = max\ C^T x$$
$$st\quad \sum_{i=1}^{n} A_j x_j \le b$$
$$\forall A_j \in K_j, \qquad j = 1, \dots n$$

Which is equivalent to:

$Q(x) = max \ C^T x$

$$st \quad \sum_{i=1}^{n} a_{ij}x_j + \sum_{j=1}^{n} \hat{a}_{ij} \leq b_i \forall i$$

With $\quad K_j = [a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}]$

$A_j \in K_j \subseteq IR^m$

Soyster (1973) equally showed that, assuming $\tilde{a}_{ij} \in K_j$, the optimal robust solution boils down to the solution of the initial linear programming problem by assigning values to the coefficients of the random variables in the matrix A as follows:

$\tilde{a}_{ij} = a_{ij} + \hat{a}_{ij}$ if the ith constraint is in the form $(\leq)$ negative

$\tilde{a}_{ij} = a_{ij} - \hat{a}_{ij}$ if the ith constraint is in the form $(\geq)$ positive

According to Minoux (2009) the uncertainty on the second member follows the robust formulation according to Soyster (1973) by integrating the vector *b* in the matrix of constraints and adding an (n+1)th variable of decision $x_{n+1} = 1$. Thus, the problem becomes:

$max \ C^T x$

$$st \quad \sum_{i=1}^{n+1} A_j x_j \leq 0 \quad \forall A_j \in K_j, \quad j = 1, \dots n,$$

$$, A_{n+1} \in K_{n+1}, \quad x_{n+1} = 1$$

Therefore, in order to build a Robust Linear Programming Model, it suffices to developed the stochastic model and then modify it following the proposal of Soyster (1973).

PROBLEM DESCRIPTION AND MODELLING

Description of the Problem

The objective here is to use the stochastic linear programming to maximize profits in a small – medium size poultry farming project while respecting customers' demands. The stochastic linear programming is useful here since most of the parameters are random, especially commands and productions. Other parameters shall be included and treated during modelling. Since it is a poultry farming project, the main objective shall be to find the number of birds of each species and age that the farmer must have at a period *t* to maximize the benefit while respecting the orders from customers.

It is worth noting that several other constraints could come into consideration out of that of command and production. These could include without restriction the constraints of money, time, and space. The farmer may have to rent more space if demand is high or may even be forced to buy from elsewhere if there is not enough to meet up with the request of the customers.

Modelling

The intension here is to build a stochastic/robust linear programming model that will maximize the profits while respecting the constraints mentioned in the problem description. Assuming that orders are made by period that is the duration of an order is a period, assume also that the species of the same age have the same weight. Suppose now that:

$x_{ia}(t)$, is the variable that represents the number of species of type i and of age *a* to be raised in the owned space during the period t to meet the command of the period t.

$s_{ia}(t)$, is the variable that represents the number of species of type i and of age *a* to be raised in the rented space during the period t to meet the command of the period t.

$st_{ia}(t)$, is the variable that represents the number of species of type i and age *a* to be raised by a subcontractor during the period t to meet the command of the period t.

$a_{max}(t)$, represents the money available to the farmer during the period t.

$e_{max}(t)$ represents the space available to the farmer during the period t.

$\delta_{max}(t)$ represents the maximum time available to the farmer during the period t.

$a_{ia}(t)$ is the money needed to care for a species of type i and of age a raised during the period t to satisfy orders of the period t.

$a'_{ia}(t)$ is the money needed to care for a species of type i and of age a raised during the period t and stored during the period $t-1$.

$e_{ia}(t)$ is the space needed to contain a species of type *i* and of age a during the period t.

$\delta_{ia}(t)$ is the time needed to take care of a species of type i and of age a during the period t.

$p_{ia}(t)$ is the number of species of type i and of age a produced by a species of type i and of age a during the period t. This parameter turns random because it is not always possible to know the number of young ones a bird can produce, thus the association with the effect of a random variable $\bar{\theta}_{ia}(t)$.

$d_{ia}(t)$ is the number of species of type i and of age a commanded during the period t. This parameter becomes random because production may not meet the command or customers may not respect their commands, hence the association with the effect of a random variable $\bar{\eta}_{ia}(t)$.

$d_{ia}(t, j)$ is the number of species of type i and of age a commanded during the period t but which will be delivered during the period $t + j$. Since this parameter is random, a random effect variable $\bar{\eta}_{ia}(t, j)$ is associated with it.

$l(t)$ is the money required for the rental of a unit of space during the period t.

During the period *t,* it could happen that customers may either still be in possession of the produce that was delivered during the period *t-1* or that are to be delivered in subsequent periods.
In such a case, let say:

$stockx_{ia}(t-1)$ be the number of species of type i and of age a remaining from the period $t-1$ which must be kept in the farmers space in order to maximise profit of the period t as well as subsequent periods.

$stocks_{ia}(t-1)$ be the number of species of type i and of age a remaining from the period $t-1$ which must be kept in the rented space in order to maximise profit of the period t as well as subsequent periods.

$stockst_{ia}(t-1)$ be the number of species of type i and of age a remaining from the period $t-1$ which must be sub contracted in order to maximise profit of the period t as well as subsequent periods.

$stock_{ia}(t-1)$ the number of species of type i and of age a remaining from the period $t-1$ which must be continued with during the

period **t**.

Since $d_{ia}(t)$ and $p_{ia}(t)$ have been declared as random variables, and it happens that production does not meet the commands for the period **t**, then let $y_{ia}(\varepsilon(t))$ be the number of species of type **i** and of age **a** that has to be bought to satisfy the command and let $q_{ia}(t)$ be the profit on a species bought to satisfy the command, then this two quantities shall be used to construct the demand function.

After describing the possible but not all different variables and parameters needed for the construction of the stochastic linear programming model, the construction of the constraints is required.

Constraints of the model

Constraint on money

The maximum money available to the farmer during the period **t** is $a_{max}(t)$. For a species of type **i** and age **a**, one could spend $a_{ia}(t)$ in some cases and $a'_{ia}(t)$ in others, thus, for $(x_{ia}(t) + stockx_{ia}(t-1))$, one would spend

$$\left[\left(a_{ia}(t)\right)(x_{ia}(t)) + (a'_{ia}(t))(stockx_{ia}(t-1))\right],$$

Therefore for **n** species of type **i** with ages ranging from 1 to **T**, the expected spending should be:

$$\sum_{i=1}^{n}\sum_{a=1}^{T}\left[\left(a_{ia}(t)\right)(x_{ia}(t)) + (a'_{ia}(t))(stockx_{ia}(t-1))\right] \qquad (1)$$

during the period t using the farmers owned space.

Similarly, let a unit of space be rented at $l(t)$, thus to raise $s_{ia}(t) + stocks_{ia}(t-1))$ in the rented space, the expected spending is $(\ e_{ia}(t)l(t) + a_{ia}(t)s_{ia}(t)) + (e_{ia}(t)l(t) + a'_{ia}(t)stocks_{ia}(t-1))$ for the rent. Analogous to the reasoning in Equation (1), one should spend in the rented space for **n** species of type **i** with ages ranging from 1 to T

$$\sum_{i=1}^{n}\sum_{a=1}^{T}(e_{ia}(t)l(t) + a_{ia}(t)s_{ia}(t)) +$$
$$(e_{ia}(t)l(t) + a'_{ia}(t)stocks_{ia}(t-1)) \qquad (2)$$

It follows that, the money to be spend by the farmer during the period **t** is the total amount of money in Equations (1) and (2). Expenditures should not exceed the maximum $a_{max}(t)$ money available during the period **t.** Thus the constraint on money can be written as:

$$\sum_{i=1}^{n}\sum_{a=1}^{T}\left[\left(a_{ia}(t)\right)(x_{ia}(t)) + (a'_{ia}(t))(stockx_{ia}(t-1))\right] +$$
$$\left[\left(e_{ia}(t)\right)l(t)\left(s_{ia}(t) + stocks_{ia}(t-1)\right)\right] \le a_{max}(t) \qquad (3)$$

Constraint on the space

The maximum space available to the farmer during the period t is $e_{max}(t)$.

To raise species of type **i** with age **a** requires $e_{ia}(t)$ units of the owned space, thus

$(x_{ia}(t) + stockx_{ia}(t-1)$, will require $\left[(e_{ia}(t))(x_{ia}(t) + stockx_{ia}(t-1))\right]$, and accordingly for **n** species of type **i** with ages ranging from 1 to **T**, the farmer would need

$$\sum_{i=1}^{n}\sum_{a=1}^{T}\left[\left(e_{ia}(t)\right)(x_{ia}(t) + stockx_{ia}(t-1))\right]$$

amount of space during the period **t** using the owned space. The use of the owned space should not exceed the maximum

space $e_{max}(t)$ available to the farmer during the period **t**. The constraint on space can therefore be written as follows:

$$\sum_{i=1}^{n}\sum_{a=1}^{T}\left[\left(e_{ia}(t)\right)(x_{ia}(t) + stockx_{ia}(t-1))\right] \le e_{max}(t) \qquad (4)$$

Constraint on the time

To care for a species of type **i** and of age **a,** $\delta_{ia}(t)$ unit of time is needed. The assumption is that, the farmer takes care of the stocks both in the owned and rented spaces. This means that it will require

$$\left(\delta_{ia}(t)\right)\left[\left((x_{ia}(t) + stockx_{ia}(t-1)) + \left(s_{ia}(t) + stocks_{ia}(t-1)\right)\right]\right.$$

time to take care of a bird. Hence for **n** species of type **i** with ages ranging from 1 to *T*, will require a total time given as follows:

$$\sum_{i=1}^{n}\sum_{a=1}^{T}\left(\delta_{ia}(t)\right)\left[\left((x_{ia}(t) + stockx_{ia}(t-1)) + \left(s_{ia}(t) + stocks_{ia}(t-1)\right)\right]\right.$$

Since the time to be used should not be more than the maximum available time $\delta_{max}(t)$ of the period **t,** it implies the constraint on time can be written as:

$$\sum_{i=1}^{n}\sum_{a=1}^{T}\left(\delta_{ia}(t)\right)\left[\left((x_{ia}(t) + stockx_{ia}(t-1)) + \left(s_{ia}(t) + stocks_{ia}(t-1)\right)\right] \le \delta_{max}(t) \qquad (5)$$

Constraint on the commands

Here the constraints, are in two folds; a constraint on the orders placed during the previous periods and the period t to satisfy periods t and later and a constraint to satisfy orders placed on the species of age zero, for periods **t** and later.

In the first case, the quantity $x_{ia}(t) + s_{ia}(t) + st_{ia}(t) + stock_{ia}(t-1)$ must equal the sum of orders placed during the previous periods, the period t and later and can be evaluated by $\sum_{j=0}^{T}(d_{ia}(t,j) + \bar{\eta}_{ia}(t,j))$. This constraint is then written as:

$$x_{ia}(t) + s_{ia}(t) + st_{ia}(t) + stock_{ia}(t-1) + y_{ia}(\varepsilon(t)) = \sum_{j=0}^{T}((d_{ia}(t,j) + \bar{\eta}_{ia}(t,j))$$

For the second constraint, $(p_{ia}(t) + \bar{\theta}_{ia}(t))$ is the number of species produced by one species of type **i** and of age **a** during the period **t**, then for $(x_{ia}(t) + s_{ia}(t) + st_{ia}(t) + stock_{ia}(t-1))$ which represents the sum of species of type **i** and of age **a** raised in the owned and rented spaces, those subcontracted and the stock of period **t-1**. The total production would be given as $(p_{ia}(t) + \bar{\theta}_{ia}(t))(x_{ia}(t) + s_{ia}(t) + st_{ia}(t) + stock_{ia}(t-1))$. Sometimes, this production might not meet the command and will therefore require $y_{i0}(\varepsilon(t))$ making up for the deficit. Since the production has to be equal to the orders of species of type **i** and that will be of age **j** at the period **t+j**, this constraint can be written as:

$$(p_{ia}(t) + \bar{\theta}_{ia}(t))(x_{ia}(t) + s_{ia}(t) + st_{ia}(t) + stock_{ia}(t-1)) + y_{ia}(\varepsilon(t)) \ge \sum_{j=0}^{T}(d_{i0}(t,j) + \bar{\eta}_{i0}(t,j)) \qquad (6)$$

Constraint on the stock

The stocks in owned, rented spaces plus the stock sub contracted

during the period *t-1*, should equal the total stock for the same period, hence the constraint is written as:

$$stockx_{ia}(t-1) + stocks_{ia}(t-1)$$
$$+stockst_{ia}(t-1) = stock_{ia}(t-1) \qquad (7)$$

The objective function

The objective function is that which will maximise profits after the model has been solved taking into consideration the identified constraints. Therefore if:

$c_{ia}(t)$: is the profit obtained on a species of type *i* and age *a* raised in owned space during the period *t*.

$c'_{ia}(t)$: is the profit obtained on species of type *i* and age *a* of the stock of the period *t-1*elevated in owned space during the period *t*.

$b_{ia}(t)$: is the profit obtained on one species of type *i* and age *a* raised in rented space during the period *t*.

$b'_{ia}(t)$: is the profit obtained on species of type *i* and age *a* of the stock of the period *t-1*elevated in the rented space during the period *t*.

$g_{ia}(t)$:is the profit obtained on species of type *i* and age *a* raised

by a subcontractor during the period *t*.

$g'_{ia}(t)$: is the profit obtained on species of type *i* and age *a* of the stock of period *t-1* raised by a subcontractor during the period *t*.

Then the total profit would be written as:

$$\sum_{i=1}^{n}\sum_{a=1}^{T}[(x_{ia}(t))c_{ia}(t) + (stockx_{ia}(t-1))c'_{ia}(t) + (s_{ia}(t))b_{ia}(t) + (stocks_{ia}(t-1))b'_{ia}(t) + (st_{ia}(t))g_{ia}(t) + (stockst_{ia}(t-1))g'_{ia}(t)] \qquad (8)$$

It could happen that the production does not satisfy the demand, which means other birds have to be bought from the market to make up the deficit.

As the number of species of type *i* and age *a* $y_{ia}(\epsilon(t))$ to pay for the deficit is a random variable, stochastic linear programming and particularly the L-Shaped method requires calculating the mathematical expectation of profit for the recourse function. As $q_{ia}(t)$ is the profit achieved on a species of type *i* and age *a* needed to make up for the deficit, the recourse function would then be given as:

$$E_{\bar{\epsilon}}(\sum_{i=1}^{n}\sum_{a=1}^{T} q_{ia}(t)y_{ia}(\varepsilon(t))) \qquad (9)$$

Therefore the objective function would be written as:

Maximize

$$\sum_{i=1}^{n}\sum_{a=1}^{T}[(x_{ia}(t))c_{ia}(t) + (stockx_{ia}(t-1))c'_{ia}(t) + (s_{ia}(t))b_{ia}(t) + (stocks_{ia}(t-1))b'_{ia}(t) + (st_{ia}(t))g_{ia}(t) + (stockst_{ia}(t-1))g'_{ia}(t)] + E_{\bar{\varepsilon}}(\sum_{i=1}^{n}\sum_{a=1}^{T} q_{ia}(t)y_{ia}(\varepsilon(t))) \qquad (10)$$

The Stochastic model

With the definition of the objective function, the Stochastic Linear Programming model for solving this problem can now be written as:

Maximize

$$\sum_{i=1}^{n}\sum_{a=1}^{T}[(x_{ia}(t))c_{ia}(t) + (stockx_{ia}(t-1))c'_{ia}(t) + (s_{ia}(t))b_{ia}(t) + (stocks_{ia}(t-1))b'_{ia}(t) + (st_{ia}(t))g_{ia}(t) + (stockst_{ia}(t-1))g'_{ia}(t)]$$
$$+ E_{\bar{\varepsilon}}(\sum_{i=1}^{n}\sum_{a=1}^{T} q_{ia}(t)y_{ia}(\varepsilon(t)))$$

Subject to the constraints

$$\sum_{i=1}^{n}\sum_{a=1}^{T}[(a_{ia}(t))(x_{ia}(t) + (a'_{ia}(t))stockx_{ia}(t-1)) + (e_{ia}(t)l(t) + a_{ia}(t))s_{ia}(t)) + (e_{ia}(t)l(t) + a'_{ia}(t))stocks_{ia}(t-1))] \le a_{max}(t)$$

$$\sum_{i=1}^{n}\sum_{a=1}^{T}[(e_{ia}(t))(x_{ia}(t) + stockx_{ia}(t-1))] \le e_{max}(t)$$

$$\sum_{i=1}^{n}\sum_{a=1}^{T}(\delta_{ia}(t))[((x_{ia}(t) + stockx_{ia}(t-1)) + (s_{ia}(t) + stocks_{ia}(t-1))] \le \delta_{max}(t)$$

$$\sum_{a=1}^{T}(p_{ia}(t)+\bar{\theta}_{ia}(t))(x_{ia}(t) + s_{ia}(t) + st_{ia}(t) + stock_{ia}(t-1)) + y_{i0}(\varepsilon(t)) = \sum_{j=0}^{T}(d_{i0}(t,j) + \bar{\eta}_{i0}(t,j))$$

$$x_{ia}(t) + s_{ia}(t) + st_{ia}(t) + stock_{ia}(t-1) + y_{ia}(\varepsilon(t)) = \sum_{j=0}^{T}((d_{ia}(t,j) + \bar{\eta}_{ia}(t,j))$$

$$stockx_{ia}(t-1) + stocks_{ia}(t-1) +stockst_{ia}(t-1) = stock_{ia}(t-1)$$
$$x_{ia}(t), y_{ia}(\varepsilon(t)) \ge 0$$

The Robust model

The robust equivalence of this problem as proposed by Soyster (1973) would be written as:

Maximize

$$\sum_{i=1}^{n}\sum_{a=1}^{T}\left[(x_{ia}(t))c_{ia}(t) + (stockx_{ia}(t-1))c'_{ia}(t) + (s_{ia}(t))b_{ia}(t) + (stocks_{ia}(t-1))b'_{ia}(t) + (st_{ia}(t))g_{ia}(t) + (stockst_{ia}(t-1))g'_{ia}(t)\right]$$
$$+ \sum_{i=1}^{n}\sum_{a=1}^{T}q_{ia}(t)y_{ia}(t))$$

Subject to the constraints

$$\sum_{i=1}^{n}\sum_{a=1}^{T}\left[(a_{ia}(t) + \hat{a}_{ia}(t))(x_{ia}(t) + (a'_{ia}(t) + \widehat{a'}_{ia}(t))stockx_{ia}(t-1)) + (e_{ia}(t)l(t) + (a_{ia}(t) + \hat{a}_{ia}(t))s_{ia}(t))\right.$$
$$\left. + (e_{ia}(t)l(t) + (a'_{ia}(t) + \widehat{a'}_{ia}(t)))stocks_{ia}(t-1))\right] \leq a_{max}(t)$$

$$\sum_{i=1}^{n}\sum_{a=1}^{T}\left[(e_{ia}(t))(x_{ia}(t) + stockx_{ia}(t-1))\right] \leq e_{max}(t)$$

$$\sum_{i=1}^{n}\sum_{a=1}^{T}(\delta_{ia}(t))[((x_{ia}(t) + stockx_{ia}(t-1)) + (s_{ia}(t) + stocks_{ia}(t-1))] \leq \delta_{max}(t)$$

$$\sum_{a=1}^{T}(p_{ia}(t) + \hat{p}_{ia}(t))(x_{ia}(t) + s_{ia}(t) + st_{ia}(t) + stock_{ia}(t-1)) + y_{i0}(t)) + \sum_{j=0}^{T}(d_{i0}(t,j) - \hat{d}_{i0}(t,j))h_{i0}(t) \geq 0$$

$$x_{ia}(t) + s_{ia}(t) + st_{ia}(t) + stock_{ia}(t-1) + y_{ia}(t) + \sum_{j=0}^{T}((d_{ia}(t,j) - \hat{d}_{ia}(t,j))h_{ia}(t) \geq 0$$

$$stockx_{ia}(t-1) + stocks_{ia}(t-1) + stockst_{ia}(t-1) = stock_{ia}(t-1)$$
$$x_{ia}(t), \ y_{ia}(\varepsilon(t)) \geq 0; \quad h_{ia}(t) = 1$$

Table 1. Represents unit cost of production for each specie in the owned and rented spaces for each period.

Space	Broilers	Gallus gallus	Period
owned space	[295 ; 475]	[200 ; 300]	1
	[855 ; 1105]	[500 ; 650]	2
Rented space	[325 ; 500]	[250 ; 350]	1
	[920 ; 1130]	[550 ; 700]	2

Table 2. Represents unit profit from each bird and specie in FCFA from all the sources.

Profit in FCFA	Broilers	Gallus gallus	Period
owned space	[500 ; 650]	[400 ; 500]	1
	[500 ; 650]	[500 ; 600]	2
Rented space	[400 ; 500]	[350 ; 450]	1
	[400 ; 500]	[400 ; 500]	2
Sub contracted	[65 ; 80]	[50 ; 60]	1
	[65 ; 80]	[50 ; 60]	2
Bought	[-200 ; 0]	[-200 ; 0]	1
	[-200 ; 0]	[-200 ; 0]	2

Table 3. Represents the quantity of space required by each species for its production.

Parameter	Broilers	Gallus gallus	Period
Space in m²	0,20*0,20	0,20*0,20	1
	0,30*0,30	0,30*0,30	2

Table 4. Represents the time required by each species for its production.

Parameter	Broilers	Gallus gallus	Period
Time in hours	1	1	1
	2	2	2

Table 5. Represents number of chicks that can be produced by each species in a given period.

Parameter	Broilers	Gallus gallus	Period
Production by Species	0	0	1
	0	[10 ;15]	2

Table 6. Represents orders made for the different species per period.

Parameter	Broilers	Gallus gallus	Period
Order by Species	100	350	1
	250	300	2

Table 7. Summary of the available resources for the first period.

Money max	300,000 FCFA
Space max	10,000 m²
Time max	39000 minutes

REAL LIFE APPLICATION

The proposed model was then tested with data collected from a poultry farm in the West Region of Cameroon. In this farm, two species of birds were raised; table birds (broilers) and traditional birds (Gallus Gallus) and this is carried out twice a year thus giving two periods for the year. Data collected was pre-processed to give the figures in Tables 1 to 7. The breeder did not give values per species, but in a global form of which it was divided by the number of birds in each species to get unit values.

Construction of the real life model

For simplicity, the model was applied to the periods of one time step. The variable x_{11}, s_{11}, st_{11} and y_{11} designate broilers of the same age of first period raised in owned space, rented space, subcontracted and bought respectively.

The variable $x_{12}, s_{12}, st_{12}, y_{12}$ designate broilers raised in the second period respectively in owned space, rented space, subcontract and bought.

The variable x_{21}, s_{21}, st_{21} and y_{21} designate Gallus gallus raised in the first period in owned space, rented space, subcontract and bought respectively.

The variable x_{22}, s_{22}, st_{22} and y_{22} designate Gallus gallus of the second period raised respectively in owned space, rented space, subcontracted and bought.

The linear program of this period can be written as follows:

Maximize $575x_{11} + 450s_{11} + 75st_{11} - 100y_{11} + 575x_{12} + 450s_{12} + 75st_{12} - 100y_{12} + 450x_{21} + 400s_{21} + 55st_{21} - 100y_{21} + 550x_{22} + 450s_{22} + 55st_{22} - 100y_{22}$

Subject to: −

$$475x_{11} + 500s_{11} + 3000y_{11} + 1105x_{12} + 1130s_{12} + 5000y_{12} + 300x_{21} + 350s_{21} + 1500y_{21} + 650x_{22} + 700s_{22} + 3000y_{22} \leq 1000000$$

$x_{11} + 2x_{12} + x_{21} + 2x_{22} \leq 10,000$
$x_{11} + s_{11} + 2x_{12} + 2s_{12} + x_{21} + s_{21} + 2x_{22} + 2s_{22} \leq 39,000$
$x_{11} + s_{11} + st_{11} + y_{11} = 100$
$x_{12} + s_{12} + st_{12} + y_{12} = 250$
$x_{21} + s_{21} + st_{21} + y_{21} = 350$
$x_{22} + s_{22} + st_{22} + y_{22} = 300$

Solving the problem using Matlab

The function ' linprog' of Matlab was used to solve the

problem in order to come out with the best decision that will maximise the profit of the poultry farmer. In other words, results returned by this function will allow the farmer to maximise profit.

After having entered the values of the various variables in the LINPROG function of MATLAB, the following results were obtained in order to maximise profit:

1) The farmer should raise in the first period 100 broilers in owned space, 250 broilers of same age should be subcontracted in the second period, while 350 Gallus

gallus should be raised in owned space in the first period, 227 Gallus gallus of the same age village should be raised in the second period in owned space and 73 Gallus gallus of the same age should be sub contracted in the second period in order to make maximal profit.

This model yielded a profit of 362,580 FCFA, which is far much greater that the profit of 241,485 FCFA that is usually made when he uses his local model made up of the following:

1) 100 broilers raised in the first period in owned space;
2) 200 broilers of the same age in the second period in owned space, 50 subcontracted;
3) 105 Gallus gallus raised in owned space in the first period and 195 subcontracted;
4) 350 Gallus gallus of the same age subcontracted in the second period.

It is obvious that while using the Linear Programming model, there will be an increase in profit of 121,095FCFA which represent a more than 50% increase in profit. This figure is highly significant and would contribute greatly in poverty alleviation especially in most African countries where majority livelihood depends on agriculture.

CONCLUSION

Stochastic Linear Programming and Robust Linear Programming are little known in the livestock sector despite their potential of optimising even in the presence of uncertainty in data. This uncertainty is a typical characteristic of most of the parameters involved in this livestock sector in particular and the poultry industry in particular. The main difficulty associated with the stochastic model is the need to provide the probability distribution functions of the underlying stochastic parameters whereas the robust model addresses the uncertain nature of the problem without making specific assumptions on probability distributions. In this article, we have been able to construct a simple stochastic model for the poultry farming problem and then convert it to a robust model by using the proposal of Soyster (1973).

The model thus constructed was tested with data obtained from a poultry farm in the Bambutus Division of the Western Region of Cameroon. Since the farmer could only give global amounts for each of the factors studied, the unit prices were obtained by dividing these global figures by the number of birds of each species being raised. The Linprog function of Matlab was used to run the program.

The results showed a greater than 50% increase in profit if the values obtained from our model were implemented as compared to the result obtained when using the locally used model. In fact our model yielded a profit of 362,580 FCFA, which is far much greater that the profit of 241,485 FCFA made when implementing the local model.

It is therefore hoped that if farmers are educated on the use of such models in the day to day running of their poultry activities, the result will not only be the improvement of livelihood on the part of the poultry farmers, but it will go a long way to satisfy the daily growing needs of poultry products by the entire community.

Future work

To the best of our knowledge, this technique has not been applied to this type of activity before, thus it is very possible that we might not have been able to exploit all the possible factors that could come into action in the attempt to obtain maximum profit in this economic activity. Therefore, many other factors could be included depending on the circumstances and the volume of the activity.

CONFLICT OF INTERESTS

The authors have not declared any conflict of interests.

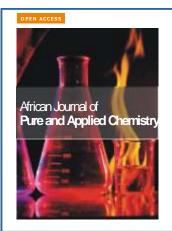## REFERENCES

Alidaee B, Kochenberger GA (2005). A note on a simple dynamic programming approach to the single-sink, fixed-charge transportation problem. Transportation Science 39(1):140-143.

Birge JR, Louveaux F (2011). Introduction to stochastic programming. Springer-Verlag, New York.

Cooper L, LeBlanc LJ (1997). Stochastic transportation problems and other network related convex problems. Naval Research Logistics Quarterly 24:327-336.

Dantzig G (1955) Linear programming under uncertainty. Management Science 1:197-206.

Delgado C (2005). Rising demand for meat and milk in developing countries: implications for grasslands-based livestock production. Grassland: a global resource pp. 29-39.

Dupacova J (1998). Reflections on robust optimization. Marti K, Kall P, eds. Stochastic Programming Methods and Technical Applications. Springer Verlag, Berlin Heidelberg pp. 111-127.

Minoux M (2009). Robust linear Programming with right-hand-side uncertainty, duality and applications. Encyclopedia of Optimization pp. 3317-3327.

Minoux M (2010). Robust network optimization under polyhedral demand uncertainty is NP-hard. Discrete Applied Mathematics 158:597-603.

Mustafa CP (2005). Robust Optimization. Bilkent University. https://www.ie.bilkent.edu.tr/~mustafap/courses/rt4.pdf

Nji A (2004). Why poor people remain poor. Key elements for poverty alleviation and sustainable development. Buma Kor Publishers. Yaoundé-Cameroon.

Powell WB, Topaloglu H (2003). Stochastic Programming in Transportation and Logistics. Handbooks in Operations Research and Management Science10:555-635.

Rosegrant et al (2009). Looking into the future for Agriculture and AKST (Agricultural Knowledge Science and Technology) in Agriculture at the Cross. Washington, DC Island Press Road pp 307-376.

Ruszczinski A, Shapiro A (2003). Stochastic Programming. Elsevier, Amsterdam.

Soyster AL (1973). Convex programming with set-inclusive constraints

and applications to inexact linear programming. Operations Research 43(2):264-281.
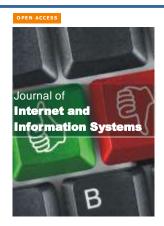
Steinfeld H, Gerber P, Wassenaar TD, Castel V, Rosales M, Rosales M, de Haan C (2006). Livestock long shadow: environmental issues and options. FAO, Rome, Italy.

Thornton PK, Jones PG, Owiyo TM, Kruska RL, Herrero M, Kristjanson P (2006). Mapping Climate vulnerability and Poverty in Africa 200 p, Nairobi (Kenya) ILRI.

Van Landeghem H, Vanmaele H (2002). Robust planning: a new paradigm for demand chain planning. Journal of Operations Management 20(6):769-783.

Yu C, Li H (2000). A robust optimization model for stochastic logistic problems. International Journal of Production Economics 64(1-3):385-397.

Ziemba WT, Vickson RG (1975), Stochastic Optimization Models in Finance. Academic Press New York.
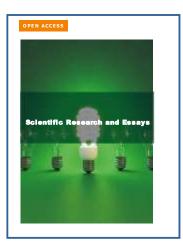
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